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Gauged motion in general relativity and in Kaluza-Klein theories

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Abstract: In a recent paper [1] a new generalization of Killing motion, the *gauged motion*, has been introduced for stationary spacetimes where it was shown that the physical symmetries of such spacetimes are well described through this new symmetry. In this article after a more detailed study in the stationary case we present the definition of gauged motion for general spacetimes. The definition is based on the gauged Lie derivative induced by a threading family of observers and the relevant reparametrization invariance. We also extend the gauged motion to the case of Kaluza-Klein theories.

Keywords: Space-Time Symmetries, Classical Theories of Gravity.

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Contents

1. Introduction

The well known definition of a spacetime symmetry is based on the concept of *isometry* and formulated as the Killing equation (or Killing motion)

$$\pounds_\xi g_{ab} = 0 \tag{1.1}$$

However applying the concepts of the *threading* approach (section 3) to the spacetime decomposition one can show that the physical symmetries of a spacetime corresponding to a given timelike family of observers might be different from its apparent mathematical symmetries exhibited by the metric of that spacetime [1]. This difference is due to the fact that these spacetime symmetries are the motions under which the observer clock rates, the spatial metric h_{ab} and the gravitoelectromagnetic fields $\mathcal{E}_a$ and $\mathcal{B}_{ab}$ are invariant and therefore there exists a non sensitivity to the gravitomagnetic potential A_a up to a suitable *gauge* transformation [1, 2]. A known example is the NUT spacetime [3], although the metric itself dose not have spherical symmetry but the spacetime really does, in fact one can show that all the curvature invariants of the spacetime are spherically symmetric [4, 5]. The same property have been shown for the cylindrical (planar) symmetry through the so called cylindrical (planar) analogue of NUT space [6, 1]. The outline of the paper is as follows. After a brief mathematical preliminary on orbit manifolds and the relevant splitting structure in the next section, we give a summary on the $1+3$ formalism of the spacetime decomposition in section 3. Concepts introduced in this section will be used extensively throughout the paper. In section 4, on the basis of a reparametrization invariance of the threading decomposition, we define a generalization of Lie derivative called the *gauged Lie derivative*. In section 5 we introduce the idea of *gauged motion* or *gauged isometry* in the stationary spacetimes and in section 6 we show that in this case the gauged isometries are in accordance with the symmetries of the curvature invariants. Section 7 is devoted to our primary motivation of introducing the idea of gauged motion i.e providing a clear and covariant manifestation of the hidden symmetries of NUT space which had been a matter of discussion for sometime. Some of the general properties of *gauged Killing vectors*, (GKV), in stationary case are discussed in section 8. In section 9 the gauged motion is defined for general spacetimes according to the ideas of sections 2, 3 and 4. Then considering Kaluza-Klein theories and symmetries of the corresponding 4-D spacetimes, an extended gauged motion is presented is section 10 for such theories. In section 11 we make a brief comparison between spacetime symmetries through gauged motion and spontaneous symmetry breaking in quantum field theory. Conclusions and further applications are summarized in section 12. There is also an appendix on the derivation of the spatial force and the geometrical interpretations of the gravitoelectromagnetic fields.

2. Orbit manifolds and the parametric structure

Let $\Re = (\mathcal{R}, +)$ the additive Lie group of real numbers, act smoothly on a (pseudo)-riemannian manifold (M, g) on the left. That is a smooth map

$$\varrho : \mathcal{R} \times M \to M \tag{2.1}$$

satisfying

$$\begin{aligned} \varrho(\tau_2 + \tau_1, q) &= \varrho(\tau_2, \varrho(\tau_1, q))\,; \quad \forall (\tau_1, \tau_2, q) \in \mathcal{R}^2 \times M \\ \varrho(0, q) &= q\,; \quad \forall q \in M \end{aligned} \tag{2.2}$$

defines a global flow on M or in other words *threads* it. Such a structure is called a (smooth) $\Re$-manifold.[1] Now the set of maps

$$\begin{aligned} \varrho_\tau :& \quad M \to M\,; \quad \forall \tau \in \mathcal{R} \\ \varrho_\tau(q) &= \varrho(\tau, q) \end{aligned} \tag{2.3}$$

[1]If M is taken to be a state space, this is the definition of a smooth dynamical system of M.

with the corresponding composition as the group operation forms a 1-parameter group of *diffeomorphisms* of M and the orbits

$$\Re.q = \{\varrho(\tau, q)\,;\,\forall\tau \in \mathcal{R}\}\,;\quad \forall q \in M \tag{2.4}$$

form a congruence of one dimensional immersed smooth submanifolds of M on M. Moreover the corresponding stabilizer of each point $q \in M$

$$\Re_q = \{\tau \in \mathcal{R} \,|\, \varrho(\tau, q) = q\} \tag{2.5}$$

is a closed subgroup of $\Re$ and the left coset space $\frac{\Re}{\Re_q}$ can be identified with $\Re.q$ [7]. Therefore due to the fact that the only closed subgroups of $\Re$ are

$$(0, +)\,,\qquad \Re\,,\qquad (\{n\tau; \forall n \in \mathcal{Z}\}, +)\,;\quad \tau \in \mathcal{R} \tag{2.6}$$

each orbit could be diffeomorphic to $\mathcal{R}$, S^1 or a point but here all the orbits are assumed to be either $\mathcal{R}$ or S^1. Denoting such an action by the triplet $(\varrho, \Re, M)$ and considering the orbit space

$$\bar{M} = \frac{M}{\Re} = \{\Re.q\,;\quad \forall q \in M\} \tag{2.7}$$

together with the quotient topology, i.e. the largest topology which makes the map

$$\Pi : M \to \bar{M}\,;\quad \Pi(q) = \Re.q \tag{2.8}$$

everywhere continuous, one may now raise the question whether $\bar{M}$ could be given the structure of a smooth manifold. The answer is in the affirmative and the conditions needs to be fulfilled by the smooth action on M are explained in the following theorem.

Theorem 1. *Let G be a Lie group acting smoothly on a smooth manifold M (on the left)*[2]*. The topological orbit space $\bar{M} = \frac{M}{G}$ has the structure of a smooth manifold with $\Pi : M \to \bar{M}$ a submersion if and only if*

$$\sim = \left\{\left(q, q'\right) \in M \times M \,|\quad \Pi(q) = \Pi\left(q'\right)\right\} \tag{2.9}$$

is a closed smooth submanifold of $M \times M$ with respect to its product topology. The manifold structure on $\bar{M}$ satisfying these requirements is then unique, moreover if M and Π are analytic, so is $\bar{M}$ [8].

In the case of a compact Lie group G, however a sufficient condition is that the action be free, i.e the stablizer of each point of M is the identity subgroup of G [9].

In what follows we assume that, apart from the analyticity, the above theorem is satisfied. All the above arguments could be simply encoded in the fact that either of the triplets $(M, \Re, \bar{M})$ or $(M, \mathrm{U}(1), \bar{M})$ is a *smooth principal bundle.*

[2]Here a manifold is defined as a Hausdorff, second countable topological space which is locally homeomorphic to a finite dimensional euclidean space.

There is a unique non-null vector field on M which generates $(\varrho, \Re, M)$. It is defined by[3]

$$\zeta : M \to TM \tag{2.10}$$

$$_q\zeta(f) = \frac{d}{d\tau}_{|\tau=0} f[\varrho_\tau(q)]\,; \quad \forall q \in M \tag{2.11}$$

where f is any real-valued function defined on a neighborhood of the point. Defining the curves

$$_q\Gamma : I = (-1,1) \to M\,; \quad \forall q \in M \tag{2.12}$$

$$_q\Gamma[\varphi(\tau)] = \varrho(\tau, q) \tag{2.13}$$

with $\varphi^{-1} : I \to \mathcal{R}$ a homeomorphism, $_q\zeta$ is the tangent vector to $_q\Gamma$ at the point q.

There is a unique *orthogonal splitting structure* on (M, g), corresponding to $(\varrho, \Re, M)$, constructed by using the projection (P) and coprojection $(P^{'})$ tensor fields, defined as

$$P^{'}(\omega, X) = -\zeta \otimes g(\omega, \zeta, X)\,; \quad \forall (X, \omega) \in TM \times T^*M \tag{2.14}$$

$$P = \delta - P^{'} \tag{2.15}$$

where

$$\delta(\omega, X) = \omega(X)\,; \quad \forall (X, \omega) \in TM \times T^*M \tag{2.16}$$

They satisfy the relations

$$P^2 = P\,; \qquad P^{'2} = P^{'}\,; \qquad PP^{'} = P^{'}P = 0\,; \qquad P + P^{'} = \delta \tag{2.17}$$

and split the tangent bundle over M to a direct sum

$$TM = {}^{\perp}TM \oplus {}^{||}TM \tag{2.18}$$

in which

$$\begin{aligned} {}^{\perp}TM &= \cup_{q \in M} {}^{\perp}T_qM \\ {}^{\perp}T_qM &= \{{}_qX \in T_qM\,; \quad {}_qX \cdot {}_q\zeta = 0\}\,; \qquad \forall q \in M \end{aligned} \tag{2.19}$$

Similarly

$$T^*M = {}^{\perp}T^*M \oplus {}^{||}T^*M\,. \tag{2.20}$$

The splittings (2.18) and (2.20) are based on the following unique decompositions

$$\begin{aligned} X &= {}^{\perp}X + {}^{||}X\,; \qquad \forall X \in TM \\ \omega &= {}^{\perp}\omega + {}^{||}\omega\,; \qquad \forall \omega \in T^*M \end{aligned} \tag{2.21}$$

where

$$\omega({}^{\perp}X) = {}^{\perp}\omega(X) = P(\omega, X)\,; \qquad \forall (X, \omega) \in TM \times T^*M\,. \tag{2.22}$$

[3]Our notation is such that given a vector field X and a point $q \in M$, $_qX \equiv X(q)$.

In this respect the projected tensor fields

$$^{\perp}T \in {}^{\perp}TM^r_s\,; \qquad \forall T \in TM^r_s \tag{2.23}$$

are defined by

$$\begin{aligned}&{}^{\perp}T\left({}^{\perp}\omega_1,\ldots,{}^{\perp}\omega_r,\ldots,{}^{\perp}X_1,\ldots,{}^{\perp}X_s\right) = T\left({}^{\perp}\omega_1,\ldots,{}^{\perp}\omega_r,\ldots,{}^{\perp}X_1,\ldots,{}^{\perp}X_s\right)\\ &\forall(\omega_1,\ldots,\omega_r,X_1,\ldots X_s) \in T^*M^r \times TM^s\,.\end{aligned} \tag{2.24}$$

Therefore the metric tensor of the orthogonal tangent bundle, $^{\perp}TM$, is

$$\bar{g} = {}^{\perp}g \tag{2.25}$$

If $\bigtriangledown$ is an affine connection on TM then

$$^{\perp}\bigtriangledown_X X' \;=\; {}^{\perp}\left(\bigtriangledown_X X'\right); \qquad \forall\left(X,X'\right) \in {}^{\perp}TM^2 \tag{2.26}$$

is an affine connection on $^{\perp}TM$ [10]. Moreover if $\bigtriangledown$ is torsion-free and compatible with g, $^{\perp}\bigtriangledown$ is torsion-free and compatible with $^{\perp}g$ where the two torsions are related to each other by [11]

$$^{\perp}\mathcal{T}\left(X,X'\right) = {}^{\perp}\left(\mathcal{T}\left(X,X'\right)\right)\;; \quad \forall\left(X,X'\right) \in {}^{\perp}TM^2\,. \tag{2.27}$$

The so called *Zelmanov* curvature tensor on $^{\perp}TM$ is defined by [12, 11]

$$\begin{aligned}&\bar{Z}\left(X',X''\right)X = {}^{\perp}\bigtriangledown_{X'}{}^{\perp}\bigtriangledown_{X''}X - {}^{\perp}\bigtriangledown_{X''}{}^{\perp}\bigtriangledown_{X'}X - {}^{\perp}\left(\pounds_{[X',X'']}X\right)\\ &\forall\left(X,X',X''\right) \in^{\perp} TM^3\end{aligned} \tag{2.28}$$

With respect to a given basis one has[4]

$$P^a_b \;=\; \delta^a_b - \frac{\zeta^a\zeta_b}{|\zeta|^2} \tag{2.29}$$

$$^{\perp}T^{a_1\ldots a_r}_{b_1\ldots b_s} \;=\; P^{a_1}_{c_1}\ldots P^{a_r}_{c_r}P^{d_1}_{b_1}\ldots P^{d_s}_{b_s}T^{c_1\ldots c_r}_{d_1\ldots d_s} \tag{2.30}$$

$$\bar{g}_{ab} \;=\; g_{ab} - \frac{\zeta_a\zeta_b}{|\zeta|^2}\,. \tag{2.31}$$

Applying the above definitions to the 4-D holonomic basis ∂_a and its dual $\mathrm{d}x^a$ one gets the 3-D, generally non-holonomic bases for the orthogonal tangent and cotangent spaces

$$^{\perp}\partial_a \;=\; P^b_a\partial_b \tag{2.32}$$

$$^{\perp}\mathrm{d}^a \;=\; P^a_d\,\mathrm{d}x^b\,. \tag{2.33}$$

Accordingly the absolute derivative corresponding to $^{\perp}\bigtriangledown$ is

$$^{\perp}\mathcal{D} \doteq {}^{\perp}\mathrm{d}^a\,{}^{\perp}\bigtriangledown_a\,. \tag{2.34}$$

Definition 1. *A* parametric orbit manifold $\bar{\mathcal{M}}$ *corresponding to an* $\Re$*-manifold* (M,g) *is the orbit manifold* $\bar{M} = \frac{M}{\Re}$ *endowed with the tensor fields and the connection defined on the orthognal tangent and cotangent bundles,* $({}^{\perp}TM, {}^{\perp}T^*M)$ *[10, 12, 13].*

[4]In this paper the Latin indices run from 0 to d-1 while the Greek ones run from 1 to d-1 for a d-manifold.

3. Summary on the $1+3$ decomposition in general relativity

Due to the coupling of space, time and matter in general relativity the true reconstruction of the 3-D space in a general spacetime has been a matter of investigation. Any satisfactory solution to this problem should respect the following two requirements.

(I) From the geometrical standpoint the definition of space has to be *intrinsic* i.e coordinate independent.

(II) From the physical standpoint since there is no notion of absolute time or absolute space in the theory of relativity, the definition has to be somehow *observer-dependent.*

To satisfy both of the above requirements one could start from a threading family of observer worldlines in a given spacetime. Then a standard experiment in general relativity (local light synchronization), based on sending and receiving light signals, defines the *spatial distance* between any two nearby observers of this family [14]. The *spatial metric tensor* defined in this way coincides (up to a minus sign) with the metric tensor ${}^{\perp}g$ of the orthogonal tangent bundle induced by the observer worldlines. According to this fact and some other similar arguments, the corresponding parametric orbit manifold is taken to be the 3-D space *realized* by the threading observers. This method is called the $1+3$ spacetime splitting or the threading decomposition [10, 15].[5] In the important special case when the worldlines are hypersurface orthogonal the result of this procedure is equivalent to the so called $3+1$ or ADM decomposition [18] in which the spacetime is foliated by hypersurfaces regarded to be the momentary spaces with the corresponding induced metrics as the momentary spatial metrics. The $3+1$ approach despite being very well known through the introduction of the total mass and energy in asymptotically flat spacetimes, suffers the deficiency of being only fairly applicable to the special case of product manifolds.

The threading decomposition leads to the following splitting of the spacetime distance element [19]

$$\mathrm{d}s^2 = \mathrm{d}T^2 - \mathrm{d}L^2 \tag{3.1}$$

where the invariants $\mathrm{d}L$ and $\mathrm{d}T$ are respectively the *spatial* and *temporal* length elements of two nearby events as measured by the threading observers. They are constructed from the normalized tangent vector $u^a = \frac{\zeta^a}{|\zeta|}$ to the timelike curves (observer 4-velocities) in the following way

$$\mathrm{d}L^2 = -\bar{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b \tag{3.2}$$
$$\mathrm{d}T = u_a\mathrm{d}x^a\,. \tag{3.3}$$

Defining

$$h = |\zeta|^2 \tag{3.4}$$
$$A_a = -\frac{\zeta_a}{|\zeta|^2} \tag{3.5}$$

[5] For various applications of this formalism refer to [16, 17].

equations (3.1) and (2.31) take the following forms

$$\mathrm{d}s^2 = h(A_a \mathrm{d}x^a)^2 - h_{ab}\mathrm{d}x^a\mathrm{d}x^b \tag{3.6}$$

$$h_{ab} = -\bar{g}_{ab} = -g_{ab} + hA_aA_b\,. \tag{3.7}$$

Given an $\Re$-manifold, a coordinate system is said to be a *preferred* one if the partial derivative operator with respect to one of the coordinates coincides locally with the ζ vector field of the action while the other coordinates label the orbits, so that[6]

$$\zeta^a \doteq (1,0,0,0)\,; \qquad A_a \doteq -\frac{g_{0a}}{g_{00}}\,; \qquad h = e^{2\nu} \doteq g_{00} \tag{3.8}$$

and the above spatial and spacetime distance elements take the following forms

$$\mathrm{d}L^2 \doteq \gamma_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \tag{3.9}$$

$$\mathrm{d}s^2 \doteq h(\mathrm{d}x^0 - A_\alpha \mathrm{d}x^\alpha)^2 - \mathrm{d}L^2 \tag{3.10}$$

where

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\,. \tag{3.11}$$

Also the equations (2.32), (2.33), (2.26) and (2.28) transform into [11]

$$^\perp\partial_\alpha \doteq \partial_\alpha + A_\alpha \partial_0 \tag{3.12}$$

$$^\perp\partial_0 \doteq 0 \tag{3.13}$$

$$^\perp\mathrm{d}^\alpha \doteq \mathrm{d}x^\alpha \tag{3.14}$$

$$^\perp\mathrm{d}^0 \doteq A_\alpha \mathrm{d}x^\alpha \tag{3.15}$$

$$^\perp\nabla_\beta X^\alpha \doteq X^\alpha_{*\beta} + \Upsilon^\alpha_{\beta\eta}X^\eta\,; \qquad \forall X \in {}^\perp TM \tag{3.16}$$

$$\bar{Z}^\alpha_{\beta\eta\rho} \doteq \Upsilon^\alpha_{\beta\rho*\eta} - \Upsilon^\alpha_{\beta\eta*\rho} + \Upsilon^\alpha_{\sigma\eta}\Upsilon^\sigma_{\rho\beta} - \Upsilon^\alpha_{\sigma\rho}\Upsilon^\sigma_{\eta\beta} \tag{3.17}$$

where

$$\Upsilon^\alpha_{\beta\eta} \doteq \frac{1}{2}\gamma^{\alpha\rho}(\gamma_{\beta\rho*\eta} + \gamma_{\rho\eta*\beta} - \gamma_{\beta\eta*\rho}) \tag{3.18}$$

and $*$ stands for $^\perp\partial$.

In this context the velocity of a particle as measured by the threading observers is given by [14]

$$\mathcal{V}^\alpha \doteq \frac{\mathrm{d}x^\alpha}{\sqrt{h}\,{}^{\parallel}\mathrm{d}^0} \tag{3.19}$$

or covariantly

$$\mathcal{V}^a = \frac{^\perp\mathrm{d}^a}{\mathrm{d}T}\,. \tag{3.20}$$

The *spatial* 4-*force* due to the spacetime curvature which deviates test particles from geodesics of the orbit manifold and makes them follow the geodesics of spacetime is defined by [14]

$$\mathcal{F}^a = \frac{^\perp\mathcal{D}\mathcal{P}^a}{\mathrm{d}T} \tag{3.21}$$

[6] Hereafter equations written in this preferred coordinate system are denoted by the sign $\doteq$.

where[7]

$$\mathcal{P}^a = \frac{m}{\sqrt{1-\mathcal{V}^2}}\mathcal{V}^a \tag{3.22}$$

and

$$\mathcal{V}^2 = h_{ab}\mathcal{V}^a\mathcal{V}^b\,. \tag{3.23}$$

As is shown in appendix A, by introducing the *gravitoelectric*[8] and *gravitomagnetic* fields [12][9]

$$\mathcal{E}_a = -\nu_{;a} - \zeta^b(\nu_{;b}A_a + F_{ba}) \doteq \left(0, \vec{\mathcal{E}}\right) \tag{3.24}$$

$$\mathcal{B}_{ab} = F_{ab} - A_{[a}F_{b]c}\zeta^c \doteq \sqrt{\gamma}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{B}^3 & -\mathcal{B}^2 \\ 0 & -\mathcal{B}^3 & 0 & \mathcal{B}^1 \\ 0 & \mathcal{B}^2 & -\mathcal{B}^1 & 0 \end{pmatrix} \tag{3.25}$$

where

$$F_{ab} = A_{[b\,;a]} \tag{3.26}$$

and

$$\vec{\mathcal{E}} = -\left(\nabla\nu + \vec{A}_{,0} + \nu_{,0}\vec{A}\right) \tag{3.27}$$

$$\vec{\mathcal{B}} = \nabla\times\vec{A} + \vec{A}\times\vec{A}_{,0} \tag{3.28}$$

$\bar{\mathcal{F}}_a = h_{ab}\mathcal{F}^b$ is found to be

$$\bar{\mathcal{F}}_a + f_a = \frac{m}{\sqrt{1-\mathcal{V}^2}}\left(\mathcal{E}_a + \sqrt{h}\,\mathcal{B}_{ab}\,\mathcal{V}^b\right) \tag{3.29}$$

where

$$f_a = \frac{1}{\sqrt{h}}\pounds_\zeta\bar{\mathcal{P}}_a = 2\frac{m}{\sqrt{1-\mathcal{V}^2}}D_{ab}\mathcal{V}^b + m_{||}\left(-\frac{1}{\sqrt{h}}\zeta^b\nu_{;b} + \zeta^cF_{cb}\mathcal{V}^b + D_{bc}\mathcal{V}^b\mathcal{V}^c\right)\mathcal{V}_a \tag{3.30}$$

$$m_{||} = \frac{m}{(1-\mathcal{V}^2)^{\frac{3}{2}}} \tag{3.31}$$

$$\bar{\mathcal{P}}_a = h_{ab}\mathcal{P}^b \tag{3.32}$$

and [12]

$$\mathcal{D}_{ab} = \frac{1}{2\sqrt{h}}\pounds_\zeta h_{ab} \tag{3.33}$$

f_a can be regarded as a *friction like force* which appears in the non-stationary cases and the right hand side of the equation (3.29) can be called the *gravito-Lorentz* force.

[7]We use gravitational units where $c = G = 1$.

[8]For reviews on the subject of Gravitoelectromagnetism see references [2] and [20].

[9]Notice that *in this paper* the signs $[,]$ and $(,)$ denote the commutation and anti-commutation over indices, in example for a tensor T_{ab}, $T_{[ab]} = T_{ab} - T_{ba}$.

Using $\gamma_{\alpha\beta}$ a the metric tensor, one can write the vacuum Einstein field equations for a general spacetime in the following *quasi-Maxwell* form [12]

$$
\begin{aligned}
{}^{\perp}\nabla.\vec{\mathcal{E}} &\doteq \frac{1}{2}h\vec{\mathcal{B}}^2 + \vec{\mathcal{E}}^2 - \hat{\mathcal{D}} - \frac{1}{\sqrt{h}}\partial_0\mathcal{D} &(3.34)\\
{}^{\perp}\nabla \times \left(\sqrt{h}\vec{\mathcal{B}}\right) &\doteq 2\left(\vec{\mathcal{E}} \times \sqrt{h}\vec{\mathcal{B}} + \vec{\mathcal{S}}\right) &(3.35)\\
\bar{Z}_{\alpha\beta} &\doteq \mathcal{E}_\alpha\mathcal{E}_\beta - \frac{1}{2}{}^{\perp}\nabla_{(\alpha}\mathcal{E}_{\beta)} - \frac{h}{2}\mathcal{B}_{\alpha\eta}\mathcal{B}_{\beta}^{\cdot\eta} + &(3.36)\\
&+\frac{\sqrt{h}}{2}\left(\mathcal{D}_{\alpha\eta}\mathcal{B}_{\beta}^{\cdot\eta} + \mathcal{B}_{\alpha\eta}\mathcal{D}_{\beta}^{\eta}\right) + 2\mathcal{D}_{\alpha\eta}\mathcal{D}_{\beta}^{\eta} - \mathcal{D}\mathcal{D}_{\alpha\beta} - \frac{1}{\sqrt{h}}\partial_0\mathcal{D}_{\alpha\beta} &(3.37)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{D} &= Tr(\mathcal{D}_{ab}) &(3.38)\\
\hat{\mathcal{D}} &\doteq \mathcal{D}_{\alpha\beta}\mathcal{D}^{\alpha\beta} &(3.39)\\
S^\alpha &\doteq {}^{\perp}\partial^\alpha\mathcal{D} - {}^{\perp}\nabla_\beta\mathcal{D}^{\alpha\beta} &(3.40)
\end{aligned}
$$

and $\bar{Z}_{\alpha\beta}$ is the 3-D Ricci tensor constructed from $\bar{Z}_{\alpha\beta\eta\rho}$. For a stationary spacetime the gravitoelectric and gravitomagnetic fields are curl-free and divergenceless respectively. Therefore by taking the preferred coordinate system to be the one adapted to the congruence of its timelike Killing vector, the field equations simplify significantly [2].

4. Gauged Lie derivative

Now we investigate the expected reparametrization invariance of the threading decomposition. Considering two vector fields ζ and $\grave{\zeta}$, the question is whether they produce the same induced $1+3$ *physics*, in the sense discussed in section 3. The answer comes in two parts, for $\grave{\zeta}$ to reproduce the same threading orbits and orthogonal splitting structure as ζ i.e.

$$\grave{\bar{M}} = \bar{M}\,; \qquad \grave{P} = P \tag{4.1}$$

we only need to have

$$\grave{\zeta} = \Omega\zeta \tag{4.2}$$

where Ω is a real-valued function on M. However if we further require that ζ and $\grave{\zeta}$ satisfy

$$
\begin{aligned}
\mathcal{E}_a &= \grave{\mathcal{E}}_a\,; \qquad \sqrt{h}\mathcal{B}_{ab} = \sqrt{\grave{h}}\grave{\mathcal{B}}_{ab} &(4.3)\\
\frac{\zeta^a}{|\zeta|} &= \frac{\grave{\zeta}^a}{|\grave{\zeta}|}\,; \qquad \mathcal{V}^a = \grave{\mathcal{V}}^a &(4.4)
\end{aligned}
$$

along with

$$\frac{h(q_1)}{h(q_2)} = \frac{\grave{h}(q_1)}{\grave{h}(q_2)}\,; \qquad \forall(q_1,q_2) \in M^2 \tag{4.5}$$

i.e yield the same spatial force and the same ratio of the proper observer clock rates at any two arbitrary points, a more restricted form of the relation (4.2) is resulted, that is

$$\grave{\zeta} = \kappa\zeta\,; \qquad \forall\kappa \in \mathcal{R}\,. \tag{4.6}$$

Now according to (4.6) the two parametrizations of the threading orbits corresponding to ζ and $\grave{\zeta}$ which defined by $\zeta = \partial_\tau$ and $\grave{\zeta} = \partial_{\grave{\tau}}$ respectively, are related to each other by

$$\grave{\tau} = \frac{1}{\kappa}\tau - \phi \tag{4.7}$$

where ϕ is any real-valued function on M satisfying the following condition

$$q_2 \in \Re.q_1 \Rightarrow \phi(q_1) = \phi(q_2)\,; \qquad \forall (q_1, q_2) \in M^2 \tag{4.8}$$

that is the values of ϕ are the same along each threading orbit and so it can be equivalently regarded as a function on $\bar{M}$.

As a conclusion the physics of the $1+3$ spacetime splitting is *quantitatively* invariant under a reparametrization of the threading orbits given by (4.7)–(4.8).

In this context given an $\Re$-manifold M and two vector fields X and Y, we define the gauged Lie derivative of Y with respect to X by following the same procedure used to define the standard Lie derivative but incorporating the previously mentioned reparametrization invariance of the threading decomposition.

Considering a point $q \in M$ and a coordinate system S, the Lie derivative of Y with respect to X is given by [21]

$$({}_q\pounds_X Y)^a = \lim_{\delta\lambda\to 0} \frac{{}_{q'}Y^a_{|S'} - {}_qY^a_{|S}}{\delta\lambda} \tag{4.9}$$

where q and q' are both on the same X-orbit with coordinates x^a and $x^a + \delta\lambda X^a$ respectively and S' is another coordinate system which is related to S by

$$x'^a = x^a - \delta\lambda X^a\,. \tag{4.10}$$

According to the above definition ${}_q\pounds_X Y^a$ measures the momentary variation rate of Y^a as seen by a coordinate frame moving along the X-orbit at the point q. Now to incorporate the invariance under the reparametrization (4.7)–(4.8) we try as an alternative to the usual definition of Lie derivative the following one

$$({}^\phi_q\pounds_X Y)^a \doteq \lim_{\delta\lambda\to 0} \frac{{}_{q'}Y^a_{|\grave{S}} - {}_qY^a_{|S}}{\delta\lambda} \tag{4.11}$$

where q' is the *same point as before*, S is a preferred coordinate system for the threading orbits and the coordinate system $\grave{S}$ is related to S by

$$\grave{x}^a \doteq x^a - \delta\lambda X^a - \zeta^a \left(\delta\lambda\phi(x^\alpha) + \delta\kappa\, x^0\right). \tag{4.12}$$

Keeping $\bar{\kappa} = \delta\kappa/\delta\lambda$ constant, the above limit yields

$${}^\phi\pounds_X Y^a = \pounds_X Y^a - \zeta^a Y^b \phi_{,b} - \bar{\kappa}\zeta^a Y^0\,. \tag{4.13}$$

However since we request the result of a gauged Lie derivative be a tensor field, we have to set $\bar{\kappa} = 0$. This choice however does not reduce the generality of our argument from physical standpoint because $\bar{\kappa}$ introduces a constant rescaling of the time coordinate. Therefore

$${}^\phi\pounds_X Y^a = \pounds_X Y^a - \zeta^a Y^b \phi_{;b} \tag{4.14}$$

defines the *gauged Lie derivative* of a vector field with respect to a doublet, a vector field and *a guage*, a real-valued function $\phi \doteq \phi(x^\alpha)$ on M .

Demanding that for any real-valued function f on M

$$ {}^{\phi}\pounds_X \mathrm{f} = \pounds_X \mathrm{f} \tag{4.15} $$

and requesting the gauged Lie derivative to respect the Leibniz rule, for a one-form ω we obtain

$$ {}^{\phi}\pounds_X \omega_a = \pounds_X \omega_a + \zeta^b \omega_b \phi_{;a} \,. \tag{4.16} $$

Similarly for tensor fields of type $(1,1)$ and $(0,2)$

$$ {}^{\phi}\pounds_X T^a_b \; = \; \pounds_X T^a_b - \zeta^a T^c_b \phi_{;c} + \zeta^c T^a_c \phi_{;b} \tag{4.17} $$

$$ {}^{\phi}\pounds_X T_{ab} \; = \; \pounds_X T_{ab} + \zeta^c T_{cb} \phi_{;a} + \zeta^c T_{ac} \phi_{;b} \,. \tag{4.18} $$

The gauged Lie derivative of other mixed tensor fields are defined in the same manner. This new definition will be used in the following sections to introduce the gauged motion.

5. Gauged motion in the stationary case

The first order variation of $\mathrm{d}s^2$ under

$$ \delta x^a = \delta\lambda \, K^a \tag{5.1} $$

corresponding to the infinitesimal motion $x^a \to x^a + \delta\lambda K^a$, generated by a vector field K^a, yields [21]

$$ \delta(\mathrm{d}s^2) = \delta\lambda \mathrm{d}x^a \mathrm{d}x^b \pounds_K g_{ab} \tag{5.2} $$

which vanishes if K is an isometry generator of the spacetime. To derive the so called gauged motion, in this case we use the same method as above but apply it to the spatial and temporal line elements $\mathrm{d}T^2$ and $\mathrm{d}L^2$ given in (3.1). The formulation should be in such a way that the reparametrization invariance (4.7)–(4.8) to be incorporated in the definition of a *physical* symmetry. To achieve this goal we demand that under the variation (5.1)[10] the following requirements hold.

(I)

$$ \delta(\mathrm{d}L^2) = 0 \tag{5.3} $$

from which we have

$$ \pounds_K h_{ab} = 0 \,. \tag{5.4} $$

(II)

$$ \delta h = 0 \tag{5.5} $$

so that

$$ \pounds_K h = 0 \tag{5.6} $$

[10] To distinguish between gauged Killing vectors and the Killing vectors we denote the former by K^a and the latter by ξ^a.

which in turn by (3.24) reduces to

$$\pounds_K \mathcal{E}_a = 0\,. \tag{5.7}$$

(III) Finally

$$^{\phi}\pounds_K A_a = 0 \tag{5.8}$$

which is demanded by the arguments of the previous section and due to which we have

$$\pounds_K A_a \;=\; \phi_{;a} \tag{5.9}$$

$$\pounds_K F_{ab} = 0 \tag{5.10}$$

where in the stationary case

$$\mathcal{B}^{\alpha} \doteq \frac{1}{2\sqrt{\gamma}}\epsilon^{\alpha\beta\eta}F_{\beta\eta}\,. \tag{5.11}$$

One can obtain the same result by demanding the following requirement

$$\delta(\mathrm{d}T) = \delta\lambda\;\mathrm{d}\phi\,; \qquad \phi_{,0} \doteq 0\,. \tag{5.12}$$

Another hint to the requirements I, II and III comes from the fact that a time transformation of the form

$$x^0 \to x'^0 \doteq x^0 - \phi(x^\alpha) \tag{5.13}$$

implies

$$A_a \to A_a + \phi_{,a} \tag{5.14}$$

while $\gamma_{\alpha\beta}$ and g_{00} remain invariant [1]. We note that in the above discussion the invariance of the gravitomagnetic fields under the gauged motion was implied *independently* of their appearance through the gravito-Lorentz force or the quasi-Maxwell form of Einstein field equations.

According to the above arguments all the physical aspects of a stationary spacetime , defined through a $(1+3)$ splitting based on its timelike isometry curves, are invariant under a special kind of spacetime motions called *gauged motions*. The generator of such a motion is called a gauged Killing vector field and is defined as follows.

Definition 2. *A* gauged Killing vector field *(GKV), in an stationary spacetime threaded by its timelike Killing vector field is a vector field K^a satisfyng*

$$\pounds_K h = 0 \tag{5.15}$$

$$\pounds_K h_{ab} = 0 \tag{5.16}$$

$$\pounds_K A_a = \phi_{;a} \tag{5.17}$$

$$\zeta^a \phi_{;a} = 0 \tag{5.18}$$

As a consequence of the equations (3.7) and (5.15)–(5.17) we have

$$\pounds_K g_{ab} = h\phi_{;(a}A_{b)}\,. \tag{5.19}$$

Now if the *fact* presented in section 9 is applied to the set of equations (5.15)–(5.18), one arrives at the following equivalent version of the above definition[11].

Definition 3. *For a stationary spacetime, a vector field K^a generates a gauged motion (*gauged isometry*) with respect to a threading family of isometry comoving observers if*

$$\pounds_K g_{ab} = h\phi_{(;a}A_{b)} \tag{5.20}$$

$$\pounds_\zeta K^a = 0 \tag{5.21}$$

along with

$$\pounds_\zeta \phi = 0\,. \tag{5.22}$$

It is notable that any Killing vector field satisfies the equations (5.20) and (5.22) with $\phi = 0$, but to be a GKV it has to satisfy the condition (5.21) as well, i.e. its components should be time-independent in the preferred coordinate system. Although it is natural to expect the Killing vector fields of a stationary spacetime respect this requirement, there are exceptions such as the boost generators of the Minkowski spacetime.

6. Curvature invariants of stationary spacetimes and the gauged motion

In this section we investigate the role of gauged motion to manifest the symmetries of the curvature invariants of a stationary spacetime. First we show that for such an spacetime in the preferred coordinate system the gauged Killing vectors are basically the same as the usual Killing vectors apart from a local shift in their time components. To prove this we start from equation (5.20) in the following form

$$g_{ab,n}K^n + g_{nb}K^n_{,a} + g_{an}K^n_{,b} = h\phi_{;(a}A_{b)}\,. \tag{6.1}$$

Now in the preferred coordinate system for ζ^a the above equation takes the form

$$g_{ab,n}K^n + g_{0b}(K^0_{,a} + \phi_{,a}) + g_{a0}(K^0_{,b} + \phi_{,b}) + g_{a\alpha}K^\alpha_{,b} + g_{\alpha b}K^\alpha_{,a} \doteq 0\,. \tag{6.2}$$

Using the fact that $g_{ab,0} \doteq 0$ and changing the variable to

$$\xi^a = K^a + \zeta^a\phi \doteq K^a + \delta^a_0\phi \tag{6.3}$$

where $\zeta^a\phi$ is the generator of the *shift of the time zero* in the preferred coordinate system, equation (6.2) reduces to

$$g_{ab,n}\xi^n + g_{am}\xi^m_{,b} + g_{mb}\xi^m_{,a} = 0 \tag{6.4}$$

which is the Killing equation with the ξ as the Killing vector. The result of the above calculation can be summarized in the following relation

$$\pounds_K g_{ab} = h\phi_{;(a}A_{b)} \Leftrightarrow \pounds_\xi g_{ab} = 0 \tag{6.5}$$

where the relation between ξ^a and K^a is given by (6.3). In other words here again we basically see the interplay between the gauge freedom in choosing the gravitomagnetic potential and shifting the time zero as mentioned in the previous section.

[11] It should be noted that there are other generalizations of the Killing motion such as the *conformal* and *homothetic* Killing motions [22].

The relation (6.3) brings the following consequences:

(I) The gauge isometry group of a stationary spacetime forms a Lie algebra by its definition, then due to the equations (5.21), (5.22) and (6.3) for any two GKVs K and $K^{'}$

$$[K, K^{'}] = [\xi, \xi^{'}] \tag{6.6}$$

so that the Lie algebra of the gauged isometry group of a stationary spacetime is the same as the Lie algebra of the corresponding isometry group. We will meet again this fact while discussing the symmetries of NUT space.

(II) **Fact 1.** *For a stationary spacetime any curvature invariant of any order is invariant under the gauged motion.*

Using the equation (6.4) and noting that for any invariant $I^{(n)}$ of order n we have

$$\pounds_{\xi^a} I^{(n)} = \pounds_{K^a+\zeta^a\phi} I^{(n)} = 0 \tag{6.7}$$

or

$$\pounds_{K^a} I^{(n)} = -\pounds_{\zeta^a\phi} I^{(n)} \tag{6.8}$$

and so

$$\pounds_{K^a} I^{(n)} = -I^{(n)}_{,a}\zeta^a\phi \doteq -I^{(n)}_{,0}\phi = 0 \tag{6.9}$$

where in the last step we used the fact that the spacetime under consideration is stationary. So in the case of stationary spactime symmetries of the curvature invariants are described through the gauged motion.

7. NUT-type spaces and their symmetries

The three known NUT-type spacetimes are

(a) NUT space

(b) Cylindrical NUT

(c) Planar NUT

These spacetimes have the common feature of being stationary solutions of vacuum Einstein equations and respectively incorporating spherical, cylindrical and planar gravitomagnetic fields in their construction. Therefore as a consequence of the fact presented in the last section, not only their gravitoelectromagnetic fields but also all their curvature invariants follow the physical symmetries demanded by the corresponding gauged motions. In the following we consider the gauged Killing vectors of NUT space as main prototype exhibiting the main features discussed in the previous sections. Indeed it was some of the peculiar features of NUT space that motivated this study in the first place. There are two different interpretations of NUT space due to Misner [4] and Bonnor [23] which differ significantly in their physical and geometrical descriptions of the metric. Misner's interpretation seems to be the dominant one specially after the rederivation of NUT space presented in [2] where the ideas of gravtioelectromagnetism play the essential role [24]. There, a redrivation of NUT solution is achieved by looking for a stationary spacetime whose spatial metric and the gravitoelectromagnetic fields respect spherical symmetry.

NUT metric has the following four linearly independent Killing vectors

$$\xi_t = \partial_t \tag{7.1}$$

$$\xi_1 = \sin\phi\partial_\theta + \cos\phi[\cot\theta\partial_\phi + 2l\csc\theta\partial_t] \tag{7.2}$$

$$\xi_2 = \cos\phi\partial_\theta - \sin\phi[\cot\theta\partial_\phi + 2l\csc\theta\partial_t] \tag{7.3}$$

$$\xi_3 = \partial_\phi \tag{7.4}$$

Now from the above Killing vectors one could obtain two strange features of NUT space which in a sense describe the same idea. First of all one could easily see that the above Killing vectors satisfy the same commutation relations as the Killing vectors of the Schwarzschild space, namely

$$[\xi_\alpha, \xi_\beta] = -\epsilon_{\alpha\beta\gamma}\xi_\gamma \tag{7.5}$$

$$[\xi_\alpha, \xi_t] = 0 \tag{7.6}$$

where $\alpha, \beta, \gamma = 1, 2, 3$. In other words the isometry group is $\mathrm{SO}(3) \times \mathrm{U}(1)$, where ξ_t generates the group $\mathrm{U}(1)$ of time translations. As none of the Killing vectors contain a term in ∂_r, the orbits of the Killing vectors lie on $r = constant$ timelike hypersurfaces. Misner has shown that these hypersurfaces are topologically 3-spheres (S^3) , in contrast to the $S^2 \times R$ topology of the $r = constant$ hypersurfaces in the Schwarzschild metric. One of the main motivation of our work was the above peculiar symmetry behaviour of NUT space, the fact that by its mathematical appearance the NUT space is an axially symmetric spacetime, but the Lie algebra of its Killing vectors suggests that it is intrinsically an spherically symmetric spacetime. In fact it is seen that the NUT type spaces are physically spherical, cylindrical and planar respectively, specially the curvature invariants follow not the apparent mathematical symmetry of the line element $\mathrm{d}s^2$ of the spacetime under consideration, but its physical symmetry defined through the gauged Killing motion.

As a conclusion we mention that all the above arguments are encoded in the fact that the gauged isometry group of NUT, cylindrical NUT and planar NUT are $\Re \times \mathrm{SO}(3)$, $\Re \times \mathrm{U}(1) \times \Re$ and $\Re \times E_2$ respectively, where the first $\Re$ is the group of time translation. It can be checked easily that the generators of each group satisfy the equations (5.20)–(5.22) for the corresponding spacetime.

8. Some general properties

Regarding

$$K_{a;b} + K_{b;a} = h\phi_{;(a}A_{b)} \tag{8.1}$$

it can be seen immediately that unlike the usual Killing vector, the gauged Killing vector has in general a non-vanishing (non-constant) *expansion* i.e.

$$\Theta = K^a{}_{;a} = \phi_{;a}A^a\,. \tag{8.2}$$

We will find below that this main difference shows up in different contexts. Using the relation

$$T_{a;bn} - T_{a;nb} = R^m{}_{abn} T_m \tag{8.3}$$

valid for any vector and the symmetries of the Riemann curvature tensor one obtains the following identity

$$(T_{a;b} - T_{b;a})_{;n} + (T_{n;a} - T_{a;n})_{;b} + (T_{b;n} - T_{n;b})_{;a} = 0 \tag{8.4}$$

Using (8.1) again, the above relation for gauged Killing vectors reads

$$(2K_{a;b} - h\phi_{;(a}A_{b)})_{;n} + (2K_{n;a} - h\phi_{;(n}A_{a)})_{;b} + (2K_{b;n} - h\phi_{;(b}A_{n)})_{;a} = 0 \tag{8.5}$$

Using equation (8.3) and rearranging terms in the above equation we obtain

$$K_{n;ba} = R_{mabn}K^m + \left[(h\phi_{;(b}A_{n)})_{;a} + (h\phi_{;(n}A_{a)})_{;b} - (h\phi_{;(a}A_{b)})_{;n}\right] \tag{8.6}$$

which is the necessary condition to be satisfied by any gauged Killing vector. Using the above equation or equation (8.3) we find

$$K^a{}_{;ba} = R_{mb}K^m + (K^a{}_{;a})_{;b} = R_{mb}K^m + (h\phi_{;a}A^a)_{;b} \tag{8.7}$$

We notice that the above equation compared with the similar equation satisfied by Killing vectors and *homothetic Killing vectors* [25] has an extra term in the right hand side which is nothing but the gradient of the *expansion* of the gauged Killing vector K^a. In analogy with the definition of Killing bivector (KBV) [25] one can define the *Gauged Killing Bivector* (GKBV) by rewriting equation (8.1) in the following form

$$Q_{ab} = 2\ K_{a;b} - h\phi_{;(a}A_{b)} \tag{8.8}$$

where

$$Q_{ab} = K_{[a;b]} \tag{8.9}$$

could be interpreted as the *test electromagnetic field* of any gauged Killing vector field. Using the above definition along with equation (8.1) one can show that Q_{ab} satisfies

$$Q^{mn}{}_{;n} = R^{mn}K_n + \Theta^{;m} - [h\phi^{;(n}A^{m)}]_{;n} = 4\pi J^m \tag{8.10}$$

where J^m is the current corresponding to the above defined *test electromagnetic field*. There are two extra contributions to this current compared with the current corresponding to the homothetic Killing bivector [26]. The first term, as expected, is the reappearance of the gradient of the expansion of the gauged Killing vector[12] and the second one is basically the divergence of the symmetric part of $K_{n;m}$ which contributes in the definition of the shear velocity corresponding to the gauged Killing vector.

[12]Note that the homothetic Killing vectors, defined by the equation $\pounds_H g_{ab} = n g_{ab}$, do have a non zero expansion but it is a constant and so its gradient does not contribute in J^m.

9. Gauged motion in general spacetimes

Up to now we have defined the gauged motion for a stationary spacetime assuming that the threading orbits are the orbits of its timelike Killing vector field. In this section we consider the general case by relaxing this constraint and let the spacetime be either non-stationary or stationary but threaded with an arbitrary congruence of timelike orbits. Such an extension becomes physically important in some cases for example when there is a non-Killing timelike threading family of orbits suggested by the nature of the spacetime itself such as the galaxy worldlines in cosmological solutions or dust worldlines in the corresponding spacetimes. We will find that the new general definition agrees with the one given in section 5 when the restrictions hold. More precisely we are looking for a definition of spacetime symmetry as realized by an arbitrary family of threading observers in a general spacetime. A definition which ensures the invariance of the quantities measured by such a family or geometrically speaking guarantees the invariance of the structure of the corresponding parametric orbit manifold under the motion generated by a vector field. Hence considering the spatial force, the distinct role played by h and h_{ab} along with the relevant reparametrization-invariance encoded in the definition of the gauged Lie derivative, we demand our definition to satisfy the following criteria.

Given a spacetime and a threading timelike vector field ζ, the spacetime respects the gauged motion generated by a vector field K^a if there exists a real-valued function on M, ϕ, such that

$$\phi(x^a) \doteq \phi(x^\alpha)\,; \tag{9.1}$$

$${}^\phi\pounds_K h = {}^\phi\pounds_K h_{ab} = {}^\phi\pounds_K \mathcal{E}_a = {}^\phi\pounds_K \sqrt{h}\mathcal{B}_{ab} = 0 \tag{9.2}$$

$${}^\phi\pounds_K \zeta^b F_{ba} = {}^\phi\pounds_K \zeta^b h_{;b} = {}^\phi\pounds_K \mathcal{D}_{ab} = 0 \tag{9.3}$$

along with the geometrical condition

$${}^\phi\pounds_K P^a_b = 0\,. \tag{9.4}$$

The last condition ensures that through gauged motion K^a maps the bundles ${}^\perp TM$ and ${}^\perp T^*M$ to themselves , that is

$$P^a_b Y^b = 0 \Rightarrow P^a_b\, {}^\phi\pounds_K Y^b = 0\,. \tag{9.5}$$

Also note that the gravitoelectromagnetic fields not only appear in the spatial force or the quasi-Maxwell equations but also have geometrical interpretations in the splitting structure, see equations(A.20) and (A.21).

Now definitions (4.14)–(4.18), equation (2.29) and the following relations

$$\zeta^a h_{ab} = \zeta^a \mathcal{E}_a = \zeta^a \mathcal{B}_{ab} = \zeta^a \zeta^b F_{ab} = \zeta^a \mathcal{D}_{ab} = 0 \tag{9.6}$$

imply

$$\pounds_K h = 0 \tag{9.7}$$
$$\pounds_K h_{ab} = 0 \tag{9.8}$$
$$\pounds_K \mathcal{E}_a = 0 \tag{9.9}$$
$$\pounds_K \mathcal{B}_{ab} = 0 \tag{9.10}$$
$$\pounds_K (\zeta^b F_{ba}) = 0 \tag{9.11}$$
$$\pounds_K (\zeta^b h_{;b}) = 0 \tag{9.12}$$
$$\pounds_\zeta \mathcal{D}_{ab} = 0 \tag{9.13}$$
$$\pounds_K P^a_b = \zeta^a \phi_{;b}\,. \tag{9.14}$$

Not surprisingly we are interested in a more compact definition of the gauged motion. The procedure to compactify the above definition is given through the following steps.

(I) Defining

$$\pounds_K A_a = X_a \tag{9.15}$$

equations (9.7), (9.8) and (3.7) yield

$$\pounds_K g_{ab} = h A_{(a} X_{b)}\,. \tag{9.16}$$

However one should notice that the equation (9.16) by itself is not equivalent to the set of equations {(9.7), (9.8), (9.15)}. The following fact determines the extra condition required to have such an equivalence.

Fact 2. *The following two sets of equations are equivalent*

$$A = \{(9.7),\ (9.8),\ (9.15)\} \tag{9.17}$$
$$B = \{(9.16),\ (9.19)\} \tag{9.18}$$

where

$$\pounds_K \zeta^a = X_b \zeta^b \zeta^a\,. \tag{9.19}$$

Proof. The first side: $A \Rightarrow B$

$$(9.8) \Rightarrow \pounds_K h_{a0} = 0 \Rightarrow h_{ab} K^b_{,0} \doteq 0 \Rightarrow P^c_b K^b_{,0} \doteq 0 \Rightarrow P^c_b (\pounds_\zeta K)^b = 0 \Rightarrow \pounds_\zeta K \parallel \zeta \tag{9.20}$$
$$\Rightarrow \pounds_\zeta K^a = \lambda \zeta^a\,. \tag{9.21}$$

But

$$(9.15) \Rightarrow X_0 \doteq A_a K^a_{,0} \doteq A_a \lambda \zeta^a \doteq -\lambda\,. \tag{9.22}$$

The second side: $B \Rightarrow A$

$$\pounds_K h = \pounds_K (\zeta^a \zeta^b g_{ab}) \doteq \pounds_K g_{00} + 2 g_{0b} \pounds_K \zeta^b \doteq -2 g_{00} X_0 + 2 g_{00} X_0 \doteq 0 \tag{9.23}$$
$$\pounds_K A_a = \pounds_K \left(\frac{-g_{ab}\zeta^b}{h}\right) \doteq -\frac{1}{h}\left(\pounds_K g_{a0} + g_{ab} \pounds_K \zeta^b\right)$$

$$
\begin{aligned}
&\doteq X_a - \frac{1}{h}(hA_a X_0 - g_{ab}K^b_{,0}) && (9.24)\\
&\doteq X_a - \frac{1}{h}(hA_a A_b K^b_{,0} - g_{ab}K^{b,0}) \doteq X_a - \frac{1}{h}h_{ab}K^b_{,0} \\
&\doteq X_a - \frac{1}{h}g_{cd}P^c_a P^d_b K^b_{,0} && (9.25)\\
&\doteq X_a + \frac{1}{h}g_{cd}P^c_a P^d_b \pounds_K \zeta^b \doteq X_a + \frac{1}{h}\lambda g_{cd}P^c_a \left(P^d_b \zeta^b\right) = X_a && (9.26)\\
\pounds_K h_{ab} &= \pounds_K(hA_a A_b - g_{ab}) = hA_{(a}X_{b)} - \pounds_K g_{ab} = 0\,. && (9.27)
\end{aligned}
$$

The condition (9.19) carries an important geometrical interpretation which could have been considered as an independent motivation to demand it in the definition of the gauged motion from the beginning. It originates from the fact that K^a as a smooth vector field locally determines its integral curves and corresponding to them defines a one parameter family of diffeomorphisms of the manifold by translating each point along its integral curve passing through that point. In this respect considering an infinitesimal translation ϵK^a and a preferred chart (O, φ) the map ϵ_K is defined by

$$
\begin{aligned}
&\epsilon_K : M \to M && (9.28)\\
&\epsilon_K(q) \doteq \epsilon_K\left[\varphi^{-1}(x^a)\right] \doteq \varphi^{-1}(x^a + \epsilon K^a)\,; \qquad \forall q \in O\,. && (9.29)
\end{aligned}
$$

Now since due to (9.19)

$$K^\alpha_{,0} \doteq 0 \tag{9.30}$$

the map ϵ_K respects the equivalence relation by which ζ-orbits define the orbit manifold $\bar{M}$, that is

$$q \sim q' \Leftrightarrow \epsilon_K(q) \sim \epsilon_K\left(q'\right)\,; \qquad \forall\left(q, q'\right) \in M^2\,. \tag{9.31}$$

This is also valid for a finite translation generated by K^a and its corresponding map ϵ_K which maps ζ-orbits (and hence threading observers) to themselves, or simply

$$\epsilon_K(\Re.q) = \Re.\epsilon_K(q)\,; \qquad \forall q \in M\,. \tag{9.32}$$

We also note that the condition (9.19) is satisfied with vanishing λ by all the groups of space symmetries (like the gauge group SO(3) in NUT space) and so it is not a very restrictive condition.

(II) Equations (9.14), (9.19) and (2.29) yield

$$X_a = \pounds_K A_a = \phi_{;a} \tag{9.33}$$

or equivalently

$${}^\phi\pounds_K A_a = 0\,. \tag{9.34}$$

Now the conditions (9.16) and (9.19) take the forms

$$
\begin{aligned}
{}^\phi\pounds_K g_{ab} &= 0 && (9.35)\\
\pounds_K \zeta^a &= 0\,. && (9.36)
\end{aligned}
$$

(III) The conditions (9.11) and (9.13) are automatically satisfied due to the equations (9.33), (9.19) and (9.8).

(IV) Recalling the definition (3.24), the conditions (9.7), (9.9), (9.11) and (9.12) imply

$$\nu_{,0}\phi_{,a} \doteq 0 \tag{9.37}$$

so that to let a gauged isometry be *non-trivial*, that is different from an isometry, we need

$$\pounds_\zeta |\zeta| = 0 \tag{9.38}$$

The above condition which can be replaced with the condition (9.9), is interesting in the sense that it is independent of the vector field K^a. It determines the families of threading observers which are allowed to be considered for having a gauged isometry. Looking to this condition in the preferred coordinate system of the threading observers , $g_{00,0} \doteq 0$, physically it says that the rate of each observer's clock as observed by him, has to be a constant. Hence the *proper clock rate* of any observer in a threading family which respects a nontrivial gauged motion has to be the same everywhere on the observer worldline.

(V) Recalling the definition (3.25) the conditions (9.10), (9.11) and (9.33) imply

$$\nabla\phi \times \vec{A}_{,0} \doteq 0 \tag{9.39}$$

that is

$$\phi_{;a} = \Omega \zeta^b F_{ba} \tag{9.40}$$

where Ω is a real-valued function on M. This condition generally puts restriction on the spacetimes capable of admitting gauged motion as it implies that the vector field $g^{ab}\zeta^c F_{cb}$ is orthogonal to the hypersurfaces of constant ϕ, an example of which is when

$$\pounds_\zeta A_a \doteq A_{a,0} = 0\,. \tag{9.41}$$

The above arguments can be summarized as follows

Definition 4. *Given a spacetime with a timelike vector field ζ, a vector field K^a is the generator of a* gauged motion (gauged isometry) *for the corresponding observers if*

$$\pounds_K g_{ab} = \phi_{(;a}\zeta_{b)} \tag{9.42}$$
$$\pounds_K \zeta^a = 0 \tag{9.43}$$
$$\pounds_\zeta \zeta_a = \Omega \phi_{;a} \tag{9.44}$$
$$\zeta^a \phi_{;a} = 0\,. \tag{9.45}$$

It is notable that if one relaxes the condition (9.45) and defines the gauged motion with a more general gauge $\phi \doteq \phi(x^a)$, then the corresponding gauged Lie derivative satisfies the same equations as (4.14)–(4.18). Further to that the requirements (9.7)–(9.13) are retained but the equation (9.14) is replaced with

$$\pounds_K P^a_b = \zeta^a \left(\phi_{;b} + A_b \phi_{;c} \zeta^c\right) \tag{9.46}$$

and one finds $\phi_{,0} \doteq 0$ or ζ is a timelike Killing vector field. Therefore the gauged motion is defined through the equations (9.42)–(9.44) and the following equation

$$\phi_{;c} \zeta^{;c} \, \pounds_\zeta g_{ab} = 0 \,. \tag{9.47}$$

Now we ask if there is an *almost stationary* sapcetime in the sense that despite being non-stationary, it is observationally stationary to the corresponding observers of a timelike vector field. The answer is negative. To Show this fact starting from

$$\zeta^a = K^a \tag{9.48}$$

the equations (9.7), (9.8) and (9.33) yield

$$g_{00,0} \doteq 0 \tag{9.49}$$

$$\gamma_{\alpha\beta,0} \doteq 0 \tag{9.50}$$

$$A_{a,0} \doteq \phi_{,a} \Rightarrow A_a \doteq t\phi_{,a}(x^\alpha) + \mathrm{f}_a(x^\alpha) \tag{9.51}$$

so that by the following redefinition of the timelike coordinate

$$t \to t' \doteq t e^{-\phi} \tag{9.52}$$

the metric takes the form of a stationary spacetime line element

$$\mathrm{d}s^2 \doteq e^{2(\nu+\phi)} \left(\mathrm{d}t' - e^{-\phi} \mathrm{f}_\alpha \mathrm{d}x^\alpha\right) - \gamma_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta \,. \tag{9.53}$$

The other question is whether in the general time-dependent case the gauged symmetries coincide with the symmetries of the curvature invariants. The answer is again negative. One can show that unlike the stationary case, in general the curvature invariants do not respect the gauged motion. As a proof, we consider the case when the generator of gauged motion, the vector field K^a, is spacelike. Then as (9.43) holds, the vector fields ζ and K can define simultaneously a timelike and a spacelike coordinate and so a coordinate system can be a preferred one for both of them. Sitting in such a frame

$$\zeta^a \doteq (1, 0, 0, 0) \tag{9.54}$$

$$K^a \doteq (0, 1, 0, 0) \tag{9.55}$$

and equations (9.42)–(9.45) are satisfied by a line element such as

$$\mathrm{d}s^2 \doteq e^{2y}(\mathrm{d}t - (x - t)\mathrm{d}y)^2 - e^t \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \tag{9.56}$$

Now if all the curvature invariants of this metric satisfy $\pounds_{K^a} I^{(n)} = 0$ each one of them must be x^1-independent in this coordinate system, but this is not the case as the Ricci scalar shows.[13] That is, despite the fact that the orbit manifold is gauged invariant, the curvature invariants of spacetime are not generally so. Despite the above fact, in the next section we show that for Kaluza-Klein theories one can generalize the idea of gauged motion to make it capable of describing symmetries of both the physical quantities and the curvature invariants of the corresponding 4-D spacetime.

10. Kaluza-Klein theories and the extended gauged motion

One way to construct the 5-D Kaluza-Klein theories [27] is to start from a triplet structure $({}^5M, g, \zeta)$, a 5-D pseudo-riemannian manifold together with a non-null vector field ζ whose orbits make a congruence of smooth curves on M. Compact or non-compact versions of the theory are built by taking ζ-orbits to be all S^1 or $\mathcal{R}$ where in the latter case the *cylinder* condition does not hold, i.e the metric components can depend explicitly on the parameter of the ζ-orbits, denoted by ι. This is so because in Space-Time-Matter (S.T.M) theories [28], from a pure geometry in 5-D, one obtains a variety of induced matter fields in 4-D besides the electromagnetic field. In both versions to identify a 4-D manifold as the ordinary spacetime, one can let each ζ-orbit collapse to a singular point and take the corresponding parametric orbit manifold to be the ordinary spacetime.[14] We denote the line element by $\mathrm{d}S^2 = g_{\hat{a}\hat{b}} \mathrm{d}x^{\hat{a}} \mathrm{d}x^{\hat{b}}$ where $x^{\hat{a}} \in \{x^0, x^1, x^2, x^3, \iota\}$. The metric tensor is usually taken to be a solution of the vacuum Einstein field equations in five dimensions with or without the cosmological constant. Considering the orthogonal splitting structure induced by $\zeta^{\hat{a}}$ and defining

$$|\zeta|^2 = \epsilon e^{2\Phi}\,; \qquad \epsilon^2 = 1 \tag{10.1}$$

$$A_{\hat{a}} = -\frac{\zeta_{\hat{a}}}{|\zeta|^2} = -\epsilon e^{-2\Phi}\zeta_{\hat{a}} \tag{10.2}$$

the corresponding projection tensor is

$$P^{\hat{a}}_{\hat{b}} = -\delta^{\hat{a}}_{\hat{b}} + \zeta^{\hat{a}} A_{\hat{b}} \tag{10.3}$$

and therefore in a preferred coordinate system for $\zeta^{\hat{a}}$ we have

$$^{\perp}\partial_a \doteq \partial_a + A_a \partial_\iota \tag{10.4}$$

$$^{\perp}\mathrm{d}^a \doteq \mathrm{d}x^a \tag{10.5}$$

$$^{\perp}\nabla_a X^c \doteq {}^{\perp}\partial_a X^c + \Upsilon^c_{ab} X^b\,; \qquad \forall X \in {}^{\perp}TM \tag{10.6}$$

$$\Upsilon^c_{ab} \doteq \frac{1}{2}\gamma^{cd}({}^{\perp}\partial_b \gamma_{ad} + {}^{\perp}\partial_a \gamma_{db} - {}^{\perp}\partial_d \gamma_{ab}) \tag{10.7}$$

and

$$\mathrm{d}S^2 \doteq \epsilon e^{2\Phi}(\mathrm{d}\iota - A_a \mathrm{d}x^a)^2 - \mathrm{d}s^2 \tag{10.8}$$

[13] Here we have used Maple tensor package. Note also that being a counter-example, the physical significance of this metric is not important here.

[14] If the manifold is diffeomorphic to a product space $\mathcal{R} \times \Sigma$, one can equally take Σ as the $4-D$ space.

where

$$ds^2 \doteq \gamma_{ab} dx^a dx^b \tag{10.9}$$

is the line element of the orbit manifold. Therefore the metric tensor of the 4-D spacetime is γ_{ab} instead of g_{ab}, a fact which was pointed out by Einstein and Bergmann by requesting the 4-D metric tensor to be invariant under a shift of ι's zero point [29].

In this respect the corresponding 4-*velocity* and 4-*force* of particles are defined respectively [30]

$$u^a = \frac{{}^{\perp}\mathrm{d}^a}{\mathrm{d}s} \doteq \frac{\mathrm{d}x^a}{\mathrm{d}s} \tag{10.10}$$

$$\mathcal{F}^a = \frac{{}^{\perp}\mathcal{D}u^a}{\mathrm{d}s} \tag{10.11}$$

where m is the rest mass of the particle and $\mathrm{d}s = \sqrt{|\mathrm{d}s^2|}$. Derivation of the above force is similar to the case given in the appendix and a straightforward calculation yields

$$\mathcal{F}^a + \Xi m \partial_\iota u^a \doteq m\left(\frac{\mathrm{d}u^a}{\mathrm{d}s} + \Upsilon^a_{bc} u^b u^c\right) \tag{10.12}$$

where

$$\Xi \doteq \frac{\mathrm{d}\iota}{\mathrm{d}s} - A_b u^b \,. \tag{10.13}$$

Assuming that the 5-D spacetime is free of non-gravitational interactions, a test particle's worldline is given by

$$\frac{\mathrm{d}^2 x^{\hat{a}}}{\mathrm{d}s^2} + \Gamma^{\hat{a}}_{\hat{b}\hat{c}} \frac{\mathrm{d}x^{\hat{b}}}{\mathrm{d}s} \frac{\mathrm{d}x^{\hat{c}}}{\mathrm{d}s} \doteq \frac{S''}{S'} u^{\hat{a}} \tag{10.14}$$

where

$$S' = \frac{\mathrm{d}S}{\mathrm{d}s}\,; \qquad S'' = \frac{\mathrm{d}S'}{\mathrm{d}s}\,. \tag{10.15}$$

Equations (10.12) and (10.14) imply

$$\mathcal{F}^a + \Xi\, m \partial_\iota u^a \doteq m\,[\,term\ 1 + \ term\ 2\ +\ term\ 3\ +\ term\ 4\,] \tag{10.16}$$

in which

$$term\,1 = -\Xi^2\, \Gamma^a_{\iota\iota} \tag{10.17}$$

$$term\,2 = -2\,\Xi\,(\Gamma^a_{\iota b} + \Gamma^a_{\iota\iota} A_b) u^b \tag{10.18}$$

$$term\,3 = (\Upsilon^a_{bc} - 2\Gamma^a_{\iota b} A_c - \Gamma^a_{\iota\iota} A_b A_c) u^b u^c \tag{10.19}$$

$$term\,4 = \frac{S''}{S'} u^a \,. \tag{10.20}$$

Rewriting the symbols $\Gamma^a_{\hat{b}\hat{c}}$ in terms of the projected variables, and using the fact

$$\partial_\iota u^a = \mathrm{d}x^a \partial_\iota \left(ds^{-1}\right) \doteq -\frac{1}{2}\gamma_{bc,\iota} u^c u^b u^a \tag{10.21}$$

we obtain

$$\bar{\mathcal{F}}_a = \gamma_{ab}\mathcal{F}^b \doteq m\left[\Xi^2 \epsilon e^{2\Phi}\Theta_a + \Xi\left(\epsilon e^{2\Phi}\bar{F}_{ab} - \gamma_{ab,\iota}\right) u^b + \left(\frac{1}{2}\Xi\gamma_{bc,\iota} u^b u^c + \frac{S''}{S'}\right) u_a\right] \tag{10.22}$$

where

$$\Theta_a = -[\Phi_{,a} + A_{a,\iota} + A_a\Phi_{,\iota}] \tag{10.23}$$

or covariantly

$$\Theta_{\hat{a}} = -\left[\Phi_{;\hat{a}} + \zeta^{\hat{c}}(F_{\hat{c}\hat{a}} + A_{\hat{a}}\Phi_{;\hat{c}})\right] \tag{10.24}$$

is the *dilaton* field strength and

$$\bar{F}_{ab} = A_{[\,b\,;a\,]} + A_{[\,a}A_{\,b],\iota} \tag{10.25}$$

or covariantly

$$\bar{F}_{\hat{a}\hat{b}} = F_{\hat{a}\hat{b}} - A_{[\,\hat{a}}F_{\hat{b}\,]\,\hat{c}}\zeta^{\hat{c}} \tag{10.26}$$

is the *extended Maxwell* tensor field.

Therefore due to similar arguments given in the previous section, the *extended gauged motion* is defined as follows

Definition 5. *A vector field $K^{\hat{a}}$ is the generator of a gauged motion in a 5-D Kaluza-Klein theory if:*

$$\pounds_K\Phi = 0 \tag{10.27}$$

$$\pounds_K h_{\hat{a}\hat{b}} = 0 \tag{10.28}$$

$$\pounds_K\Theta_{\hat{a}} = 0 \tag{10.29}$$

$$\pounds_K\bar{F}_{\hat{a}\hat{b}} = 0 \tag{10.30}$$

$$\pounds_K\Xi = 0 \tag{10.31}$$

$$\pounds_K\frac{S''}{S'} = 0 \tag{10.32}$$

along with

$$\pounds_K A_{\hat{a}} = \phi_{;\hat{a}}\,. \tag{10.33}$$

One can again obtain a more compact version of the above definition as before which we leave it to the intrested reader.

11. Comparison with the case in QFT

As we know one can find the force-free equation of motion (geodesic equation) by the following variational principle

$$\delta\int \mathrm{d}s = 0 \tag{11.1}$$

in other words ds or its squared plays the role of a test particle lagrangian. Now in some of the solutions of Einstein field equations in general relativity we have cases such as the NUT-type spaces in which the lagrangian, due to the presence of the gravomagnetic potential, has a smaller symmetry group than the resulting physical states (described through the gravitoelectromagnetic fields). This is in contrast to the case of spontaneous symmetry breaking in QFT where states respect a smaller symmetry group than the one presented in the lagrangian . To shed more light on the comparison consider the case of a test particle moving in the field of a massive object presented by the line element $\mathrm{d}s^2$. As the test particle only feels the gravitomagnetic fields and the corresponding extra forces it will respect the geometrical symmetries imposed by them. Although this may not be a good comparison as in one side we have a classical theory and on the other side an intrinsically quantum mechanical theory, nevertheless one should not ignore the possibility that, in the case of gravity, this character may be retained in the final theory of *quantum gravity.*

12. Conclusions

A new definition of spacetime symmetry in general relativity which is properly applicable to spacetimes endowed with a *preferred* timelike vector field is presented in a covariant form. Such a preference could be suggested geometrically from symmetry considerations or physically from the energy-momentum tensor content. The definition has the advantage of manifesting some of the hidden physical symmetries of these spacetimes.

The extended version of this definition in higher dimensional spacetimes with a preferred non-null vector field, presented here for the case of 5-D Kaluza-klein theories, could be expected to find application in the various methods of quantizing gravity for the following two reasons.

Firstly the spacetime splitting procedure used here to define the gauged motion is not restricted to the case of globally product manifolds and this is important in the sense that in quantum gravity the spacetime topology is expected to be a dynamical entity.

Secondly by its definition gauged motion focuses on the physical aspects of the spacetime something reminiscent of the idea of observables in quantum field theories, further this new definition is more flexible than the standard one by admitting a gauge *freedom.*

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A. Gravitoelectromagnetic fields, the spatial force and geometry of splitting

According to the definitions (3.21) and (3.22)

$$\mathcal{F}^\alpha \doteq \frac{m}{\sqrt{h}\ {}^{||}\mathrm{d}^0}\,{}^{\perp}\mathcal{D}\frac{\mathcal{V}^\alpha}{\sqrt{(1-\mathcal{V}^2)}} \tag{A.1}$$

in which

$${}^{||}\mathrm{d}^0 \doteq \mathrm{d}x^0 - A_\beta \mathrm{d}x^\beta \doteq \frac{\mathrm{d}s}{\sqrt{h}\sqrt{(1-\mathcal{V}^2)}}\,. \tag{A.2}$$

Using the definition (2.34)

$$^{\perp}\mathcal{D}\mathcal{P}^{\alpha} \doteq \mathrm{d}x^{\beta}\left({}^{\perp}\partial_{\beta}\mathcal{P}^{\alpha} + \Upsilon^{\alpha}_{\beta\eta}\mathcal{P}^{\eta}\right) \tag{A.3}$$

so that

$$\mathcal{F}^{\alpha} + \frac{1}{\sqrt{h}}\mathcal{P}^{\alpha}_{,0} \doteq \sqrt{(1-\mathcal{V}^2)}\frac{\mathrm{d}}{\mathrm{d}s}\frac{m\mathcal{V}^{\alpha}}{\sqrt{1-\mathcal{V}^2}} + \Upsilon^{\alpha}_{\beta\eta}\frac{m\mathcal{V}^{\beta}\mathcal{V}^{\eta}}{\sqrt{1-\mathcal{V}^2}}\,. \tag{A.4}$$

Now due to the fact that particles follow the spacetime geodesics and also by the definition (3.19)

$$u^{\alpha} \doteq \frac{\mathcal{V}^{\alpha}}{\sqrt{1-\mathcal{V}^2}} \tag{A.5}$$

$$u^{0} \doteq \frac{1}{\sqrt{1-\mathcal{V}^2}}\left(\frac{1}{\sqrt{h}} + A_{\beta}\mathcal{V}^{\beta}\right) \tag{A.6}$$

replacing $\mathrm{d}u^{\alpha}/\mathrm{d}s$ by $-\Gamma^{\alpha}_{bc}u^{b}u^{c}$ yields

$$\mathcal{F}^{\alpha} + \frac{1}{\sqrt{h}}\mathcal{P}^{\alpha}_{,0} \doteq \frac{m}{\sqrt{1-\mathcal{V}^2}}[\text{term 1} + \text{term 2} + \text{term 3}] \tag{A.7}$$

where

$$\text{term 1} = -\frac{\Gamma^{\alpha}_{00}}{h} \tag{A.8}$$

$$\text{term 2} = -\frac{2}{\sqrt{h}}(\Gamma^{\alpha}_{0\beta} + \Gamma^{\alpha}_{00}A_{\beta})\mathcal{V}^{\beta} \tag{A.9}$$

$$\text{term 3} = -(\Gamma^{\alpha}_{\beta\eta} + \Upsilon^{\alpha}_{\beta\eta} + 2\Gamma^{\alpha}_{0\beta}A_{\eta} + \Gamma^{\alpha}_{00}A_{\beta}A_{\eta})\mathcal{V}^{\beta}\mathcal{V}^{\eta}\,. \tag{A.10}$$

Now using the relations

$$g_{\beta\eta} \doteq -\gamma_{\beta\eta} + hA_{\beta}A_{\eta} \tag{A.11}$$

$$g^{\beta\eta} \doteq -\gamma^{\beta\eta} \tag{A.12}$$

$$g^{0\beta} \doteq -\gamma^{\beta\eta}A_{\eta} \doteq -\bar{A}^{\beta} \tag{A.13}$$

and rewriting the quantities Γ^{α}_{bc} in terms of the spatial metric, the gravitomagnetic vector potential and their derivatives one finds that the term 3 vanishes and

$$\begin{aligned}\mathcal{F}^{\alpha} + \frac{1}{\sqrt{h}}\mathcal{P}^{\alpha}_{,0} \doteq \frac{m}{\sqrt{1-\mathcal{V}^2}}\Big[&-\left(\frac{h^{\alpha}}{2h} + \gamma^{\alpha\beta}A_{\beta,0} + \frac{h_{,0}}{2h}\bar{A}^{\alpha}\right) + \\ &+\left(\sqrt{h}\left(A_{\beta}^{;\alpha} + \bar{A}^{\alpha}A_{\beta,0} - \gamma^{\alpha\eta}A_{\eta,\beta} - \gamma^{\alpha\eta}A_{\eta,0}A_{\beta}\right) - \right. \\ &\left. - \frac{1}{\sqrt{h}}\gamma^{\alpha\eta}\gamma_{\eta\beta,0}\right)\mathcal{V}^{\beta}\Big]\end{aligned} \tag{A.14}$$

contracting $\mathcal{F}^{\alpha}$ with $\gamma_{\mu\alpha}$ one obtains

$$\bar{\mathcal{F}}_{\mu} + f_{\mu} = \frac{m}{\sqrt{1-\mathcal{V}^2}}\left[\mathcal{E}_{\mu} + \left(\vec{\mathcal{V}} \times \sqrt{h}\vec{\mathcal{B}}\right)_{\mu}\right] \tag{A.15}$$

where [15]

$$\mathcal{E}_\mu = -(\nu_{,\mu} + A_{\mu,0} + \nu_{,0} A_\mu) \tag{A.16}$$
$$\mathcal{B}^\alpha \doteq \epsilon^{\alpha\beta\eta}\ {}^\perp\partial_\beta A_\eta \doteq {}^\perp\nabla \times \vec{A} \doteq \frac{1}{2}\epsilon^{\alpha\beta\eta}\ \mathcal{B}_{\beta\eta} \tag{A.17}$$

and

$$f_\mu \doteq \frac{1}{\sqrt{h}}\mathcal{P}_{\mu,0} \tag{A.18}$$
$$\doteq \frac{m}{\sqrt{1-\mathcal{V}^2}}\frac{1}{\sqrt{h}}\gamma_{\mu\beta,0}\mathcal{V}^\beta + m_{||}\left(A_{\beta,0}\mathcal{V}^\beta + \frac{1}{2\sqrt{h}}\gamma_{\alpha\beta,0}\mathcal{V}^\alpha\mathcal{V}^\beta - \frac{1}{\sqrt{h}}\nu_{,0}\right)\mathcal{V}_\mu \tag{A.19}$$

is the extra force, a characteristic of non stationary spacetimes. The above quantities can be defined covariantly as (3.24), (3.25), (3.29), (3.30) and (3.33).

It is notable that the gravitoelectromagnetic fields can be defined (both physically and geometrically) independently from their appearance in the spatial force. Physically, since $\mathcal{E}_a$ and $\frac{\sqrt{h}}{2}\mathcal{B}_{ab}$ are the accerelation and the angular velocity tensor fields of ζ-observers respectively and geometrically, through the following equations [12]

$$\left[{}^\perp\partial_a, {}^\perp\partial_b\right] \doteq \mathcal{B}_{ab}\partial_0 \tag{A.20}$$
$$\left[{}^\perp\partial_a, \frac{1}{\sqrt{h}}\partial_0\right] \doteq \mathcal{E}_a\partial_0\,. \tag{A.21}$$

Hence a non-zero $\mathcal{B}_{ab}$ implies that the basis ${}^\perp\partial_a$ does not define a coordinate basis, a fact which also reflects itself in the following argument.

The orbits of ζ are hypersurface orthogonal if and only if there are two real-valued functions f and Ω on M such that

$$A_a = \frac{\zeta_a}{|\zeta|^2} = \Omega \mathrm{f}_{,a} \tag{A.22}$$

where the hypersurfaces satisfy $\mathrm{f}(x^a)$ =constant. An equivalent necessary and sufficient condition for (A.22) is

$$A_{[a}\nabla_b A_{c]} = 0\,. \tag{A.23}$$

Now regarding the definition (3.25)

$$A_{[a}\nabla_b A_{c]} = A_a\mathcal{B}_{bc} + A_b\mathcal{B}_{ca} + A_c\mathcal{B}_{ab} \tag{A.24}$$
$$A_{[0}\nabla_b A_{c]} \doteq \mathcal{B}_{ab} \tag{A.25}$$

so that

$$A_{[a}\nabla_b A_{c]} = 0 \Leftrightarrow \mathcal{B}_{ab} = 0\,. \tag{A.26}$$

Therefore a vector field is hypersurface-orthogonal if and only if the corresponding gravitomagnetic field vanishes.

[15] $\epsilon^{\alpha\beta\eta}$ is the 3D-permutation pseudo tensor times the factor $1/\sqrt{\gamma}$, correspond to which the vector product and the curl operator are defined as usual.

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